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A NOTE ON THE VALENCE OF CERTAIN MEANS

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ABSTRACT

Given two functions $f(z)$, $g(z)$ in the (usual) class S, we can form the new functions (arithmetic and geometric mean functions)

$$F(z) = \alpha f(z) + \beta g(z) \quad \text{and} \quad G(z) = z(f(z)/z)^{\alpha}(g(z)/z)^{\beta},$$

where α, $\beta \in (0,1)$ and $\alpha + \beta = 1$. This paper determines the maximum valence of the functions F and G.

1. Introduction

Let A denote the class of functions $f(z)$ regular in the unit disk E and $f(0) = f'(0) - 1 = 0$. Furthermore, let S, ST, SP and CC denote the subclasses of A consisting of univalent, starlike, spiral-like and close-to-convex functions respectively; then, as is well known, $ST \subset SP \subset S$ and $CC \subset S$.

In [3], Goodman proved that if functions $f(z)$ and $g(z)$ are selected properly from S, then the sum

$$F(z) = \frac{1}{2}(f(z) + g(z))$$

and the product

$$G(z) = \sqrt{f(z)g(z)}$$

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have valence infinity in E. In [4], Goodman discussed further the more general means

$$F(z) = \alpha f(z) + \beta g(z), \tag{1}$$

$$G(z) = f^{\alpha}(z) g^{\beta}(z) = z\left(f(z)/z\right)^{\alpha}\left(g(z)/z\right)^{\beta}, \tag{2}$$

where $\alpha > 0$, $\beta > 0$ and $\alpha + \beta = 1$. He proved that if

$$1/(1+e^{\pi}) < \alpha,\ \beta < e^{\pi}/(1+e^{\pi}),$$

then there are functions $f(z)$ and $g(z)$ in S such that the functions $F(z)$ and $G(z)$, defined by (1) and (2) respectively, have valence infinity in E. But [1], if $0 < \alpha \le 1/(1+e^{\pi})$, what can be said? Is there some bound on the valence of $F(z)$ and $G(z)$ that is a function of α? This note determines the maximum valence of $F(z)$ and $G(z)$, hence answers these questions.

2. Arithmetic means

We first investigate the maximum valence of $F(z)$ when $f(z)$ and $g(z)$ are in SP.

THEOREM 1: *If $\alpha > 0$, $\beta > 0$ and $\alpha + \beta = 1$, then there are functions $f(z)$ and $g(z)$ in SP such that the function $F(z)$, given by (1), has valence infinity in E.*

Proof: (i) If $\alpha = \beta = 1/2$. We define functions

$$f(z) = z(1+z)^{-1+i} \quad \text{and} \quad g(z) = z(1+z)^{-1-i}$$

where all powers are the principal branches. We see easily that $f(z)$ and $g(z)$ are in A. Then we observe that

$$\mathrm{Re}[e^{\pi i/4} z f'(z)/f(z)] > 0 \quad \text{and} \quad \mathrm{Re}[e^{-\pi i/4} z g'(z)/g(z)] > 0$$

$(z \in E)$ and hence the functions $f(z)$ and $g(z)$ are in SP.

By (1), we obtain

$$F(z) = \frac{1}{2} z\left((1+z)^{-1+i} + (1+z)^{-1-i}\right).$$

The condition $(1+z)^{-1+i} + (1+z)^{-1-i} = 0$ leads to

$$(1+z)^{-1+i}/(1+z)^{-1-i} = -1 \tag{3}$$

or $2i \ln(1+z) = (1+2n)\pi i$. We set $z_n = e^{(1+2n)\pi/2} - 1$ $(n = -1, -2, \ldots)$. For each negative integer n, z_n is in E and is the root of the equation (3), i.e., there are infinitely many points $z_n \in E$ $(n = -1, -2, \ldots)$ such that $F(z_n) = 0$.

(ii) Suppose that α and β satisfy the conditions of Theorem 1 and $\alpha \neq \beta$. Since α and β are symmetric, without any loss of generality, we may assume $\alpha < \beta$. For each pair α, β that satisfies the conditions of Theorem 1 and $\alpha < \beta$, there are infinitely many integers k such that

$$0 < \frac{1}{4k\pi} \ln \frac{\beta}{\alpha} < 1.$$

We take any integer k_1 from these integers k and set

$$b = \frac{1}{4k_1\pi} \ln \frac{\beta}{\alpha} \quad \text{and} \quad a = 1 - \sqrt{1-b^2}.$$

It is obvious that $0 < a,\ b < 1$ and $|1 - a - bi| = 1$. We define the functions

$$f(z) = z(1+z)^{-a-bi} \quad \text{and} \quad g(z) = z(1+z)^{-a+bi}$$

where all powers are the principal branches. Obviously, $f(z)$ and $g(z)$ are in A. We obtain

$$zf'(z)/f(z) = \left(1 + (1-a-bi)z\right)/(1+z). \tag{4}$$

We set $R(z) = \left(1 + (1-a-bi)z\right)/(1+z)$. Now $R(z)$ is a linear (Möbuis) transformation. It maps the unit circle onto straight line L which passes through the origin and makes an angle θ_1 $(-\pi/2 < \theta_1 < 0)$ with the positive real axis. Hence, by $R(0) = 1$, we obtain that $R(z)$ maps the unit disk onto the half-plane on the right of the line L. We set $\theta = -\pi/2 - \theta_1$. Then we have $-\pi/2 < \theta < 0$ and

$$\mathrm{Re}\,[e^{i\theta} zf'(z)/f(z)] > 0$$

$(z \in E)$. Hence $f(z)$ is in SP. Similarly, we can obtain that $g(z)$ is in SP. By (1), we have

$$F(z) = z\left(\alpha(1+z)^{-a-bi} + \beta(1+z)^{-a+bi}\right).$$

The condition $\alpha(1+z)^{-a-bi} + \beta(1+z)^{-a+bi} = 0$ leads to

$$(1+z)^{-a+bi}/(1+z)^{-a-bi} = -\frac{\alpha}{\beta} \tag{5}$$

or $2bi\ln(1+z) = \ln(\alpha/\beta) + (1+2m)\pi i$. We set $z_m = e^{d_m} - 1$,

$$d_m = \frac{1}{2b}[i\ln(\beta/\alpha) + (1+2m)\pi] \quad (m = -1, -2, \ldots).$$

Since

$$b = \frac{1}{4k_1\pi}\ln(\beta/\alpha),$$

we obtain that for each negative integer m, $z_m \in E$ and z_m is the equation (5), i.e., there are infinitely many points $z_m \in E$ such that $F(z_m) = 0$. This completes the proof of Theorem 1. ∎

Since $SP \subset S$, by Theorem 1, we obtain

COROLLARY 1: *If $\alpha > 0$, $\beta > 0$ and $\alpha + \beta = 1$, then there are functions $f(z)$ and $g(z)$ in S such that the function $F(z)$, given by (1), has valence infinity in E.*

3. Geometric means

We first give one lemma. It is due to Campbell and Singh [2].

LEMMA: *Let $G(z)$ be analytic in E and N be a simply connected region in E with $\partial N \cap \partial E = e^{i\theta}$. Let $P(z)$ and $Q(z)$ be analytic in E and satisfy*

(a) $G(z) = P(z)Q(z)$.

(b) $\lim P(z) = c \neq 0$ as $z \to e^{i\theta}$ within N.

(c) $Q(z)$ has no asymptotic values within N at $e^{i\theta}$, i.e., for every path ν in N ending at $e^{i\theta}$, $Q(z)$ does not tend to a finite or infinite limit as $z \to e^{i\theta}$ along ν.

(d) w_1 is in the range set of $Q(z)$ on N, i.e., for every $r > 0$, there is a point z in $\{|z - e^{i\theta}| < r\} \cap N$ with $Q(z) = w_1$.

(e) w_1 is not in the cluster set of $Q(z)$ on ∂N, i.e., there is no sequence z_n on ∂N for which $Q(z_n) \to w_1$.

Then cw_1 is in the range set of $G(z)$ on $\bar{N}$.

Remark: Let the region M contain the region N and $\partial M \cap \partial N = \{d\}$ be a single point set. We replace the unit disk E by M and replace $e^{i\theta}$ by the point d. The Lemma is valid.

THEOREM 2: *If $\alpha \geq \beta > 0$ and $\alpha + \beta = 1$, then there are functions $f(z)$ in SP and $g(z)$ in CC such that the function $G(z)$, given by (2), has valence infinity in E.*

Proof: Since $f(0) = g(0) = 0$, the functions $f^{\alpha}(z)$ and $g^{\beta}(z)$ are not well-defined in E, but the condition $\alpha + \beta = 1$ assures us the product $G(z)$ can be defined so that it is regular in E. If we make a cut in E by deleting the points $z \geq 0$, then each factor can be defined so that it is regular in the cut unit disk E^*. The following considerations are then valid in E^*.

We define functions

$$f(z) = z(1+z)^s \quad \text{and} \quad g(z) = -(1+z)\ln(1-z)$$

where

$$s = -\frac{\beta}{\alpha} + \sqrt{\frac{\beta}{\alpha}\left(2 - \frac{\beta}{\alpha}\right)}i, \quad z \in E$$

and power and logarithm are the principal branches. Obviously, $f(z)$ and $g(z)$ are in A.

Applying the same techniques used in Theorem 1, we can obtain that the function $f(z)$ is in SP since $|1+s| = 1$ and

$$zf'(z)/f(z) = [1 + z + sz]/(1+z).$$

We obtain easily that the nth coefficient of $g(z)$ is

$$b_n = (2n-1)/[n(n-1)], \quad n \geq 2.$$

Then we observe that $b_2 > 1$ and

$$(n+1)b_{n+1} - nb_n = -1/[(n-1)n].$$

By [5, Vol. II, page 28, problem 8], we have $g(z)$ is in CC.

From (2), we obtain

$$G(z) = f^{\alpha}(z)g^{\beta}(z) = z^{\alpha}\left(-\ln(1-z)\right)^{\beta}(1+z)^{q}$$

where

$$q = \alpha\sqrt{\frac{\beta}{\alpha}\left(2 - \frac{\beta}{\alpha}\right)}i.$$

We set

$$P(z) = z^{\alpha}\left(-\ln(1-z)\right)^{\beta}, \quad Q(z) = (1+z)^{q}, \quad w_1 = 1,$$

$c = (-1)^{\alpha}(-\ln 2)^{\beta}$, $e^{i\theta} = -1$, $M = E^*$ and N is the region bounded by a triangle with vertices -1, $-\frac{1}{2}(1+i)$ and $\frac{1}{2}(-1+i)$. Hence, by the Lemma and the remark, we obtain that $G(z)$ assumes the value $(-1)^{\alpha}(-\ln 2)^{\beta}$ infinitely often in E^*, i.e., $G(z)$ has valence infinity in E. ∎

COROLLARY 2: *If $\alpha > 0$, $\beta > 0$ and $\alpha + \beta = 1$, then there are functions $f(z)$, $g(z)$ in S such that the function $G(z)$, given by (2), has valence infinity in E.*

Proof: We note that α and β are symmetric. Since $SP \subset S$ and $CC \subset S$, hence, by Theorem 2, we obtain the Corollary. ∎

References and further results on the valence of sums and products can be found in Goodman [5] Volume II, Chapter 14, Sections 2 and 6. An interesting modification of the problem is to find the largest disk in which $F(z)$ and $G(z)$ are always univalent, when $f(z)$ and $g(z)$ are arbitrary functions from certain fixed sets.

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